DISTANCES BETWEEN NON–SYMMETRIC CONVEX BODIES AND THE MM^*-ESTIMATE.

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Abstract. Let K, D be n-dimensional convex bodes. Define the distance between K and D as

$$d(K, D) = \inf\{\lambda \mid TK \subset D + x \subset \lambda \cdot TK\},$$

where the infimum is taken over all $x \in \mathbb{R}^n$ and all invertible linear operators T. Assume that 0 is an interior point of K and define

$$M(K) = \int_{S^{n-1}} \|\omega\|_K \, d\mu(\omega),$$

where μ is the uniform measure on the sphere. We use the difference body estimate to prove that K can be embedded into $\mathbb{R}^n$ so that

$$M(K) \cdot M(K^\circ) \leq Cn^{1/3} \log^a n$$

for some absolute constants C and a. We apply this result to show that the distance between two n-dimensional convex bodies does not exceed $n^{4/3}$ up to a logarithmic factor.

1. Introduction.

The question of estimating the Banach – Mazur distance between two n-dimensional convex symmetric bodies (i.e. n-dimensional Banach spaces) is one of the central in the Local Theory. The upper estimate follows from a theorem of John: the distance between any convex symmetric body and the ellipsoid does not exceed $\sqrt{n}$. So, the distance between two such bodies is bounded by n. In 1981 Gluskin [Gl] proved that this estimate is essentially exact. More precisely, let $m \geq cn$ and let $g_1(\omega), \dots, g_m(\omega)$ be independent Gaussian vectors in $\mathbb{R}^n$. Put

$$\Gamma(\omega) = \text{abs conv } (g_1(\omega), \dots, g_m(\omega)).$$

Then with probability close to 1

$$d(\Gamma(\omega_1), \Gamma(\omega_2)) \geq cn.$$

Here and later C, c etc. mean absolute constants whose value may change from line to line.

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The situation is entirely different if one considers general convex bodies, which are not necessary symmetric. Since for such bodies the origin plays no special role, the definition of the distance should be modified to allow shifts. Namely, the distance between n-dimensional convex bodies K and D is

$$d(K, D) = \inf\{\lambda \mid TK \subset D + x \subset \lambda \cdot TK\},$$

where the infimum is taken over all $x \in \mathbb{R}^n$ and all invertible linear operators T.

By the theorem of John the distance between a convex body and an ellipsoid does not exceed n. This estimate is exact: the distance between the simplex and the ball is exactly equal to n (actually the simplex is the only convex body having this property [P]). So applying the John theorem twice we show that the distance between two convex bodies is bounded by n^2. The lower estimate is very far from this bound. No examples of non–symmetric bodies the distance between which is greater than Cn are known. Moreover, Lassak [L] proved that if one of the bodies is symmetric then the distance is bounded by $2n - 1$, while if one of the bodies is a simplex, it is bounded by $n + 2$. For $m \geq cn$ one can define a non–symmetric analog of a Gluskin body by

$$K(\omega) = \mathrm{conv}(g_1(\omega), \ldots, g_m(\omega)).$$

Then with probability close to 1

$$K(\omega) \supset \frac{c}{\sqrt{n}} B_2^n.$$

This means that the distance between two such bodies is less than $c^2 n$.

It will be shown below that the distance estimate is related to the MM^*-estimate. To formulate this more precisely we have to introduce some notation. Denote by d_K the Banach–Mazur distance between K and the Euclidean ball. Let μ be the normalized uniform measure on the sphere and let

$$M(K) = \int_{S^{n-1}} \|\omega\|_K \, d\mu(\omega), \qquad M^*(K) = M(K^\circ).$$

Here $\|\cdot\|_K$ means the Minkowski functional of K and $K^\circ = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \leq 1 \text{ for all } y \in K\}$ is the polar of K. Denote by g a standard Gaussian vector in $\mathbb{R}^n$. Define the ℓ-functional by

$$\ell(K) = \mathbb{E} \, \|g\|_K \, .$$

An easy and well known computation shows that

$$\frac{1}{\sqrt{n}} \cdot \ell(K) \leq M(K) \leq \beta_n \frac{1}{\sqrt{n}} \cdot \ell(K),$$

where $\beta_n \to 1$ when $n \to \infty$. Denote also by K_x a shift of K:

$$K_x = K - x.$$

For a convex symmetric body B consider

$$R(B) = \inf_T M(TB) \cdot M^*(TB),$$

where the infimum is taken over all invertible *affine* transformations T. This quantity plays a fundamental role in the Local Theory [M-S], [P]. In particular, proofs of the Quotient Subspace Theorem, Inverse Santalo and Brunn–Minkowski inequalities are based upon the following estimate [P, p.20]

$$R(B) \leq C \log d_B.$$

This estimate was proved by Pisier. His proof uses a result of Figiel and Tomczak–Jaegermann [F-T-J], stating that $R(B)$ is controlled by the norm of the Rademacher projection in $L_2(\mathbb{R}^n, \|\cdot\|_B)$.

This approach cannot be generalized to the non–symmetric convex bodies, since the norm of the Rademacher projection can be much bigger. For example for an n-dimensional simplex Δ_n this norm is at least $n/2$, while

$$c \log n \leq R(\Delta_N) \leq C \log n.$$

Using a modified definition of the Rademacher projection, Banasczyk, Litvak, Pajor and Szarek [B-L-P-S] proved that for any n-dimensional convex body K

$$R(K) \leq C\sqrt{n}.$$

We shall consider a different approach here. We shall inscribe a given body into some convex symmetric body for which we have a good MM^*-estimate and compare M and M^* using the comparison of the volumes. Using the volume estimates for the sections of the difference body [R] we obtain the following results.

Theorem 1. *Any n-dimensional convex body K may be embedded in $\mathbb{R}^n$, so that for every $\varepsilon > 0$ there exists a subspace $E \subset \mathbb{R}^n$ of dimension at least $(1-\varepsilon)n$ such that*

$$M^*(K) \leq 1$$

and

$$M(K \cap E) \prec \frac{1}{\varepsilon} \cdot \log^2 d_K.$$

We will write $X \prec Y$ if there exist absolute constants C, a such that $X \leq C \cdot Y \cdot (\log Y)^a$.

Remarks.

1. The Quotient Subspace theorem for convex symmetric bodies is based on the MM^*-estimate. However, the proof of it shows that it is enough to have an MM^*-estimate only for a subspace of a small codimension, as in Theorem 1.
2. An estimate similar to Theorem 1 was also proved by Litvak and Tomczak-Jaegermann [L-T-J] by a different method. Namely, they proved that under the assumptions of Theorem 1 there exists a subspace E such that

$$M(K \cap E)\, M(K^\circ) \leq C(\varepsilon) R(K-K) \cdot R(K \cap -K) \leq C'(\varepsilon) \log^2 (1 + d_K),$$

where $C(\varepsilon)$ and $C'(\varepsilon)$ depend on ε only.

Theorem 2. *Any n-dimensional convex body K may be embedded in $\mathbb{R}^n$, so that*

$$M(K)M^*(K) \prec n^{1/3}.$$

The proof of Theorems 1 and 2 consists of several steps. First, we show in Section 2 that the symmetrization of a convex body by taking it absolutely convex hull does not affect its volume significantly. As a byproduct of this observation we obtain the existence of an M-ellipsoid for general convex bodies. Then in Section 3 we prove that two bodies whose volumes are close possess a section on which their ℓ-functionals are close.

In Section 4 we use this fact to prove that the MM^*-problem can be reduced to the estimate of the volumes of sections of the difference body. By the difference body of a given convex body K we mean

$$K - K = \{x - y \mid x, y \in K\}.$$

Then to prove Theorem 1 we apply the following result [R].

Theorem 3. *Let $K \subset \mathbb{R}^n$ be a convex body and let $F \subset \mathbb{R}^n$ be an m-dimensional subspace. Then*

$$\mathrm{vol}\left((K - K) \cap F\right) \le C^m \varphi^m(m, n) \cdot \sup_{x \in \mathbb{R}^n} \mathrm{vol}\left(K \cap (F + x)\right),$$

where

$$\varphi(m, n) = \min\left(\frac{n}{m}, \sqrt{m}\right).$$

Since $\varphi(m, n) \le n^{1/3}$, we have the following immediate Corollary, which will be used to prove Theorem 2.

Corollary 1. *Let $K \subset \mathbb{R}^n$ be a convex body and let $F \subset \mathbb{R}^n$ be an m-dimensional subspace. Then*

$$\mathrm{vol}\left((K - K) \cap F\right) \le \left(C \cdot n^{1/3}\right)^m \cdot \sup_{x \in \mathbb{R}^n} \mathrm{vol}\left(K \cap (F + x)\right).$$

Also in Section 4 we derive from Theorem 1 the Quotient Subspace Theorem for general covex bodies. More precisely, applying Theorem 1 and an iteration argument similar to [P, Chapter 8], we obtain the following

Theorem 4. *Let K be a convex body in $\mathbb{R}^n$. Then for any $\varepsilon > 0$ there exist linear subspaces $E_2 \subset E_1 \subset \mathbb{R}^n$ such that $\dim E_2 > (1 - \varepsilon)n$ and for a body $D = P_{E_2}(K \cap E_1)$ one has*

$$d_D \prec \frac{1}{\varepsilon^2}.$$

Remark. It follows from the proof of Theorem 4 that the position of the origin is not important here. In particular, one can assume that the origin coincides with the center of mass of K.

In Section 5 we show that the distance between two convex bodies can be estimated in terms of MM^*. More precisely, we prove

Theorem 5. *Let K and D be n-dimensional convex bodies. Then*

$$d(K,D) \le Cn \cdot \sqrt{\log n} \cdot \big(M(K)M^*(K) + M(D)M^*(D)\big).$$

Combinig Theorems 2 and 5, we obtain the following

Corollary 2. *Let K and D be n-dimensional convex bodies. Then*

$$d(K,D) \prec n^{4/3}.$$

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2. Volume estimates.

Recall that for any convex body K there is a unique point $s \in K$ such that for any point z in the interior of K

$$\mathrm{vol}\,(K_x^\circ) \le \mathrm{vol}\,(K_z^\circ)\,.$$

This point is called the Santalo point of K.

Lemma 1. *Let K be an n-dimensional convex body and suppose that 0 is the Santalo point of K. Then*

$$\left(\frac{\mathrm{vol}\,(\mathrm{conv}\,(K,-K))}{\mathrm{vol}\,(K \cap (-K))}\right)^{1/n} \le C.$$

Proof. Using a difference body inequality due to Rogers and Shephard [R-S], we obtain that

$$\left(\frac{\mathrm{vol}\,(\mathrm{conv}\,(K,-K))}{\mathrm{vol}\,(B_2^n)}\right)^{1/n} \le \left(\frac{\mathrm{vol}\,(K-K)}{\mathrm{vol}\,(B_2^n)}\right)^{1/n} \le C \cdot \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(B_2^n)}\right)^{1/n}.$$

Now applying consecutively Santalo [S, p. 421] Rogers–Shephard and inverse Santalo [P, p. 100] inequalities, we show that the quantity above does not exceed

$$C \cdot \left(\frac{\mathrm{vol}\,(K^\circ)}{\mathrm{vol}\,(B_2^n)}\right)^{-1/n} \le C \cdot \left(\frac{\mathrm{vol}\,(K^\circ + (-K^\circ))}{\mathrm{vol}\,(B_2^n)}\right)^{-1/n} \le C \cdot \left(\frac{\mathrm{vol}\,(K \cap (-K))}{\mathrm{vol}\,(B_2^n)}\right)^{1/n}.$$

□

The existence of an M-ellipsoid for a convex body immediately follows from Lemma 1. More precisely, for any convex body $K \subset \mathbb{R}^n$ there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ such that

$$\left(\frac{\mathrm{vol}\,(K+\mathcal{E})}{\mathrm{vol}\,(K\cap\mathcal{E})} \cdot \frac{\mathrm{vol}\,(K^\circ+\mathcal{E}^\circ)}{\mathrm{vol}\,(K^\circ\cap\mathcal{E}^\circ)}\right)^{1/n} \le C$$

This remarkable result was proved by Milman for symmetric bodies. If K is non-symmetric, assume that 0 is the Santalo point of K and let $\mathcal{E}$ be an M-ellipsoid of $D = K \cap (-K)$. Then Lemma 1 implies that the covering number

$$N(K - K, D) \le 2^n \frac{\mathrm{vol}\,(K - K)}{\mathrm{vol}\,(D)} \le C^n.$$

So,

$$\begin{aligned}\left(\frac{\mathrm{vol}\,(K + \mathcal{E})}{\mathrm{vol}\,(K \cap \mathcal{E})}\right)^{1/n} &\le \left(\frac{\mathrm{vol}\,((K - K) + \mathcal{E})}{\mathrm{vol}\,(K \cap (-K) \cap \mathcal{E})}\right)^{1/n} \\ &\le \left(\frac{N((K - K), D) \cdot \mathrm{vol}\,((K \cap (-K)) + \mathcal{E})}{\mathrm{vol}\,(K \cap (-K) \cap \mathcal{E})}\right)^{1/n} \le C.\end{aligned}$$

By the inverse Santalo inequality, we have

$$\begin{aligned}\left(\frac{\mathrm{vol}\,(K^\circ + \mathcal{E}^\circ)}{\mathrm{vol}\,(K^\circ \cap \mathcal{E}^\circ)}\right)^{1/n} &\le \left(\frac{\mathrm{vol}\,((K \cap (-K))^\circ + \mathcal{E}^\circ)}{\mathrm{vol}\,((K - K)^\circ \cap \mathcal{E}^\circ)}\right)^{1/n} \\ &\le C \left(\frac{\mathrm{vol}\,((K - K) + \mathcal{E})}{\mathrm{vol}\,(K \cap (-K) \cap \mathcal{E})}\right)^{1/n} \le C.\end{aligned}$$

Remark. Lemma 1 and the existence of the M-ellipsoid were independently proved by Milman and Pajor [M-P].

3. M-ESTIMATES FOR SMALL SECTIONS.

To prove Theorems 1 and 2 we shall compare M and M^* of a general convex body with those of certain symmetric bodies. First notice that if

$$B = K - K, \qquad D = K \cap (-K),$$

then

$$\frac{1}{2} M^*(B) \le M^*(K) \le M^*(B).$$

By duality we have

$$\frac{1}{2} M(D) \le M(K) \le M(D).$$

We shall linearly embed B and thus K into $\mathbb{R}^n$ so that

$$M(B) M^*(B) \le R(B).$$

Now it is enough to compare $M(B)$ with the minimum of $M(K_x)$ over all shifts.

By Lemma 1 the volume of D is of the same order as that of B. However the volume estimate alone is not enough to conclude that $M(D)$ and $M(B)$ are of the same order. Indeed, consider the following convex symmetric bodies $B, D \subset \mathbb{R}^{n+1}$:

$$B = B_2^n + [-e_{n+1}, e_{n+1}], \qquad D = B_2^n + [-2^{-n} e_{n+1}, 2^{-n} e_{n+1}].$$

Then

$$\left(\frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(D)}\right)^{1/(n+1)} = 2^{n/(n+1)} < 2.$$

However,

$$M(B) \leq 1,$$

while

$$M(D) = \frac{1}{\sqrt{n}} \cdot \mathbb{E}\,\|g\|_D \geq \frac{1}{\sqrt{n}} \cdot \mathbb{E}\,\|\gamma e_{n+1}\|_D = \sqrt{\frac{2}{\pi}} \cdot \frac{2^n}{\sqrt{n}},$$

where γ is a normal random variable.

Nevertheless, it turns out that two convex symmetric bodies of approximately the same volume possess a large section on which their M-s are close enough. More precisely, we prove the following

Lemma 2. *Let $D \subset B$ be m-dimensional convex symmetric bodies, such that*

$$\left(\frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(D)}\right)^{1/m} \leq A.$$

Then for any $a < 1$ there exists a subspace $E \subset \mathbb{R}^m$, $dim E \geq am$, such that

$$M(D \cap E) \leq \frac{C}{\sqrt{a}} \cdot R(D) \cdot A \cdot M(B) \cdot \Big(CA \cdot M(B) \cdot M^*(B)\Big)^{a/(1-a)}.$$

Remark. Later we shall choose a so that $\Big(CA \cdot M(B) \cdot M^*(B)\Big)^{a/(1-a)} \leq C.$

Proof. Let S be an operator such that

$$M(SD) \leq R(D), \qquad M^*(SD) = 1.$$

Without loss of generality, we may assume that S is self–adjoint. Let $T = id + M^*(D) \cdot S$. Then

$$\begin{aligned} &M(TD) \leq \frac{M(SD)}{M^*(D)} \leq \frac{R(D)}{M^*(D)}, \\ &M^*(TD) \leq M^*(D) + M^*(D) \cdot M^*(SD) = 2M^*(D). \end{aligned} \tag{1}$$

By the inequality of Urysohn [P, p.6]and the inverse Santalo inequality, we get

$$\begin{aligned} &\left(\frac{\mathrm{vol}\,(TD)}{\mathrm{vol}\,(B_2^m)}\right)^{1/m} \leq M^*(TD) \\ &\left(\frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(B_2^m)}\right)^{-1/m} \leq c \cdot \left(\frac{\mathrm{vol}\,(B^\circ)}{\mathrm{vol}\,(B_2^m)}\right)^{1/m} \leq c \cdot M(B). \end{aligned}$$

Hence,

$$\begin{aligned} (\det T)^{1/m} = \left(\frac{\mathrm{vol}\,(TD)}{\mathrm{vol}\,(D)}\right)^{1/m} &= \left(\frac{\mathrm{vol}\,(TD)}{\mathrm{vol}\,(B_2^m)} \cdot \frac{\mathrm{vol}\,(B_2^m)}{\mathrm{vol}\,(B)} \cdot \frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(D)}\right)^{1/m} \leq \\ &2M^*(D) \cdot M(B) \cdot A. \end{aligned}$$

Since all the eigenvalues of T are greater than 1, there exists an invariant subspace E of T of dimension at least am, such that

$$\|T|_E\| \leq \Big(2M^*(D) \cdot M(B) \cdot A\Big)^{\frac{1}{1-a}}. \tag{2}$$

Recall that g_E be the standard Gaussian vector in E. We have

$$\begin{aligned} M(D \cap E) \leq & \frac{1}{\sqrt{\dim E}} \cdot \mathbb{E}\, \|g_E\|_D = \frac{1}{\sqrt{\dim E}} \cdot \mathbb{E}\, \|T|_E\, g_E\|_{TD} \\ & \leq \|T|_E\| \cdot \frac{1}{\sqrt{\dim E}} \cdot \mathbb{E}\, \|g\|_{TD} \leq \frac{\sqrt{m}}{\sqrt{\dim E}} \cdot \|T|_E\| \cdot M(TD). \end{aligned}$$

Then (1) and (2) imply that

$$\begin{aligned} M(D \cap E) \leq & \frac{1}{\sqrt{a}} \cdot \Big(2M^*(D) \cdot M(B) \cdot A\Big)^{\frac{1}{1-a}} \cdot \frac{R(D)}{M^*(D)} \\ \leq & \frac{C}{\sqrt{a}} \cdot R(D) \cdot A \cdot M(B) \cdot \Big(CA \cdot M(B) \cdot M^*(D)\Big)^{a/(1-a)}. \end{aligned}$$

The statement of Lemma 2 follows now from $M^*(D) \leq M^*(B)$. $\square$

Later it will be more convenient to use $\ell(K)$ and $\ell(K^\circ)$ instead of $M(K)$ and $M^*(K)$. With these parameters the statement of Lemma 2 can be reformulated as follows:

$$\ell(D \cap E) \leq C \cdot R(D) \cdot A \cdot \ell(B) \cdot \Big(CA \cdot \ell(B) \cdot \ell(B^\circ)/m\Big)^{a/(1-a)}. \tag{3}$$

Lemma 3. *Let Q be a convex body and let B be convex symmetric body in $\mathbb{R}^m$. Assume that $Q \subset B$ and*

$$\left(\frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(Q)}\right)^{1/m} \leq A.$$

Then there exists a subspace $F \subset \mathbb{R}^m$,

$$dim\ F \geq \frac{m}{\log(A \cdot \ell(B) \cdot \ell(B^\circ)/m)}$$

and a shift $Q_x = Q - x$ such that

$$\ell(Q_x \cap F) \leq CA \cdot \log d_Q \cdot \ell(B).$$

Proof. Let $y \in Q$ be a point such that the minimum for the distance from $Q - u$ to an ellipsoid with the center at the origin is attained for $u = y$ and let z be a Santalo point of Q. Put $x = (y + z)/2$ and denote $D = Q_x \cap (-Q_x)$. Put $x = (y + z)/2$. Then

$$d_{Q_x \cap (-Q_x)} \leq 3d_Q$$

and by Lemma 1

$$\left(\frac{\operatorname{vol}(B)}{\operatorname{vol}(Q_x \cap (-Q_x))}\right)^{1/m} \le \left(\frac{\operatorname{vol}(B)}{\operatorname{vol}(Q)}\right)^{1/m} \cdot \left(\frac{\operatorname{vol}(Q)}{\operatorname{vol}(Q_x \cap (-Q_x))}\right)^{1/m} \le CA.$$

Put

$$a = \frac{1}{\log\Big(A \cdot \ell(B) \cdot \ell(B^\circ)/m\Big)}$$

and apply (3). Since

$$\Big(A \cdot \ell(B) \cdot \ell(B^\circ)/m\Big)^{a/(1-a)} \le C,$$

we have that

$$\ell(Q_x \cap F) \le \ell(D \cap F) \le CR(D) \cdot A \cdot \ell(B).$$

The statement of the Lemma follows now from a theorem of Pisier, since

$$R(D) \le C \log d_D \le C \log d_Q. \quad \square$$

To prove Theorem 2 we need also the following estimate for small dimensions.

Lemma 4. *Let K be an n-dimensional convex body. For any subspace $F \subset \mathbb{R}^n$ there exists a shift K_u such that*

$$\ell(K_u \cap F) \le 2(\dim F)^2 \cdot \ell(K - K).$$

Proof. Let $\dim F = m$ and let $e_1, \dots, e_m$ be an orthonormal basis of F. Denote $F_j = \operatorname{span}(e_j)$. We have that

$$\ell((K-K) \cap F) = \mathbb{E}\, \|P_F g\|_{K-K} \le \sum_{j=1}^{m} \mathbb{E}\, \|P_{F_j} g\|_{K-K} = \sqrt{\frac{2}{\pi}} \sum_{j=1}^{m} \|e_j\|_{K-K}.$$

Notice that $\|e_j\|_{K-K} = 1/\max \|x - y\|_2$, where the maximum is taken over all $x, y \in K$ such that $x - y$ is parallel to e_j. Suppose that the maximum above is attained for the points $x_j, y_j \in K$ and put $u_j = 1/2(x_j + y_j)$. Then

$$\|e_j\|_{K-u_j} \le \frac{1}{\|x_j - u_j\|_2} = 2 \cdot \frac{1}{\|x_j - y_j\|_2} = 2 \cdot \|e_j\|_{K-K}.$$

Let $u = 1/m \sum_{j=1}^m u_j$. Then $K_u \cap F_j \supset 1/m\,(K_{u_j} \cap F_j)$, so

$$\begin{aligned} \ell(K_u \cap F) &\le \sum_{j=1}^{m} \ell(K_u \cap F_j) \le m \sum_{j=1}^{m} \ell(K_{u_j} \cap F_j) = m \sum_{j=1}^{m} \sqrt{\frac{2}{\pi}}\, \|e_j\|_{K-u_j} \\ &\le 2m \sum_{j=1}^{m} \sqrt{\frac{2}{\pi}}\, \|e_j\|_{K-K} \le 2m^2 \cdot \ell((K-K) \cap F). \quad \square \end{aligned}$$

4. Proof of the MM^* estimates.

Assume that the body K is embedded into $\mathbb{R}^n$ so that

$$\ell(K-K) \le R(K-K) \le C\sqrt{n}\log d_{K-K} \le C\sqrt{n}\log d_K \tag{4}$$

and

$$\ell((K-K)^\circ) \le \sqrt{n}. \tag{5}$$

Since $\ell(K_x) = \mathbb{E}\sup_{y\in K_x}\langle g, y\rangle$ is independent of x for $x \in \text{Int }(K)$, the last inequality means that for any x

$$\ell(K_x^\circ) \le \sqrt{n}.$$

To prove theorems 1 and 2 we combine Theorem 3 with the following

Proposition. *Let K be an n-dimensional convex body and let $\varepsilon > c\log n/n$. Let $A \le n$ and assume that for any linear subspace $F \subset \mathbb{R}^n$, $\dim F = m \ge \varepsilon n$ there exists a shift K_x such that*

$$\left(\frac{\text{vol}\,((K-K)\cap F)}{\text{vol}\,(K_x\cap F)}\right)^{1/m} \le A.$$

Suppose that K is embedded into $\mathbb{R}^n$ so that the inequalities (4) and (5) hold. Then there exists a subspace E, $\dim E \ge (1-\varepsilon)n$ and a shift K_u such that

$$\frac{1}{\sqrt{n}}\cdot\ell(K_u\cap E) \prec A\cdot\log^2 d_K\cdot\log^4 1/\varepsilon.$$

Proof. Let $t \in (\varepsilon, 1)$ and define

$$\varphi(t) = \min_{x,E}\ell(K_x\cap E), \tag{6}$$

where the minimum is taken over all interior points of K and all linear subspaces E of dimension at least $n\cdot(1-t)$. We prove the following

Claim. *Let*

$$a = \frac{1}{\log(A\log d_K/\varepsilon)}$$

Then

$$\varphi^{1/2}((1-a)t) \le \varphi^{1/2}(t) + \left(C\sqrt{n}A\log^2 d_K\right)^{1/2}.$$

To prove the Claim, choose x and E, $\dim E = n\cdot(1-t)$ so that the minimum in (6) occurs for them. By assumption there exists a shift $y \in K$ such that

$$\left(\frac{\text{vol}\,((K-K)\cap E^\perp)}{\text{vol}\,(K_y\cap E^\perp)}\right)^{1/nt} \le A.$$

Put $B = (K-K)\cap E^\perp$, $Q = K_y\cap E^\perp$. Then

$$\ell(B) \le \ell(K-K) \le \sqrt{n} \quad\text{and}\quad \ell(B^\circ) \le \ell((K-K)^\circ) \le C\sqrt{n}\log n.$$

Applying Lemma 3 we obtain another shift K_z and a subspace $F \subset E^{\perp}$ such that

$$\begin{aligned}\ell(K_z \cap F) &\leq CA \log d_Q \cdot \ell(B) \leq CA \log d_K \cdot \ell(K - K) \\ &\leq CA \log^2 d_K \cdot \sqrt{n}.\end{aligned}$$

Here

$$\begin{aligned}\dim F &\geq \frac{\dim E^{\perp}}{\log(A \cdot \ell(B) \cdot \ell^*(B)/\dim E^{\perp})} \geq \frac{nt}{\log(A \cdot Cn \log d_K/nt)} \\ &\geq \frac{nt}{\log(A \log d_K/\varepsilon)} = ant.\end{aligned}$$

Since $A \leq n$ and $t > \varepsilon > c \log n/n$, we have $\dim F \geq 1$.

For $\mu \in (0,1)$ let $u = \mu x + (1-\mu)z$. Then $K_u \supset \mu K_x$ and $K_u \supset (1-\mu)K_z$, so

$$\ell(K_u \cap E) \leq \mu^{-1} \cdot l(K_x \cap E) = \mu^{-1} \cdot \phi(t)$$

and

$$\ell(K_u \cap F) \leq (1-\mu)^{-1} l(K_x \cap F) \leq (1-\mu)^{-1} \cdot CA \log^2 d_K \cdot \sqrt{n}.$$

Since

$$\ell(K_u \cap (E \oplus F)) \leq \ell(K_u \cap E) + \ell(K_u \cap F)$$

and dim $(E \oplus F) \geq n(1-t) + atn = n \cdot (1-(1-a)t)$, we have that for any $\mu \in (0,1)$

$$\varphi((1-a)t) \leq \mu^{-1}\varphi(t) + (1-\mu)^{-1} \cdot CA \log^2 d_K \cdot \sqrt{n}.$$

Denote

$$v = \varphi(t), \qquad w = CA \log^2 d_K \cdot \sqrt{n}. \tag{7}$$

Choose now $\mu \in (0,1)$ so that

$$\theta(\mu) = \mu^{-1} v + (1-\mu)^{-1} w$$

will be minimal. For $\mu_0 = (1 - \sqrt{w/v})/(1 - w/v)$ we have

$$\theta(\mu) = (\sqrt{v} + \sqrt{w})^2. \tag{8}$$

Substituting (7) into (8), we get the statement of the Claim. $\square$

Now we complete the proof of the Proposition. Iterating the Claim we obtain that for any l

$$\varphi^{1/2}((1-a)^l) \leq \varphi^{1/2}(1) + l \cdot (CA \log^2 d_K \cdot \sqrt{n})^{1/2},$$

provided that $(1-a)^l \geq \varepsilon$. Choose l so that $(1-a)^l \geq \varepsilon \geq (1-a)^{l+1}$. Then for some subspace $\tilde{E}, \dim \tilde{E} \geq n(1-\varepsilon)$ and for some shift K_v we have

$$\ell(K_v \cap \tilde{E}) \leq \varphi((1-a)^{l+1}) \leq C \cdot l^2 \cdot (CA \log^2 d_K \cdot \sqrt{n}).$$

Since

$$l \le c\frac{\log 1/\varepsilon}{\log(1-a)} \le Ca^{-1}\log 1/\varepsilon \le C\log(A\log d_K)\cdot\log^2 1/\varepsilon,$$

we have that

$$\frac{1}{\sqrt{n}}\cdot\ell(K_v\cap\tilde{E}) \prec A\cdot\log d_K\cdot\log^4 1/\varepsilon. \quad \square$$

Proof of Theorem 1. We shall combine Theorem 3 and the Proposition. By Theorem 3 for any $m \le n$ and any m-dimensional subspace F there exists a shift K_x such that

$$\left(\frac{\mathrm{vol}\,((K-K)\cap F)}{\mathrm{vol}\,(K_x\cap F)}\right)^{1/m} \le C\frac{n}{m}.$$

So, by the Proposition, there exists a subspace $E,\ \dim E \ge (1-\varepsilon)n$ and a shift K_u such that

$$\frac{1}{\sqrt{n}}\ell(K_u\cap E) \prec \varepsilon^{-1}\cdot\log^2 d_K\cdot\log^4\varepsilon^{-1} \prec \varepsilon^{-1}\cdot\log^2 d_K.$$

We shall show that the point u can be chosen independently of the dimension.

For $1 \le l \le L = \log_2 n$ put $\varepsilon_l = 2^{-l}$. Obviously, it is enough to prove Theorem 1 for $\varepsilon = \varepsilon_l, l = 1,\dots,L$. By the Proposition we can find a shift K_{u_l} and a subspace $E_l,\ \dim E_l \ge (1-2^{-l})n$ such that

$$\frac{1}{\sqrt{n}}\ell(K_{u_l}\cap E_l) \prec 2^l\cdot\log^2 d_K.$$

Define

$$v = \frac{1}{s}\sum_{l=1}^{L}\frac{u_l}{l^2},$$

where $s = \sum_{l=1}^{L} 1/l^2$. Then $v \in K$. The same argument as before yields that for any l

$$\frac{1}{\sqrt{n}}\ell(K_v\cap E_l) \prec l^2\cdot 2^l\cdot\log^2 d_K \prec 2^l\cdot\log^2 d_K. \quad \square$$

Proof of Theorem 2. We shall use Corollary 1 to estimate the sections of the difference body. For any linear subspace $F \subset \mathbb{R}^n$ there exists a shift K_x such that

$$\left(\frac{\mathrm{vol}\,((K-K)\cap F)}{\mathrm{vol}\,(K_x\cap F)}\right)^{1/m} \le A = Cn^{1/3}.$$

Applying the Proposition with $\varepsilon = C\log n/n$, we can find a subspace $E,\ \dim E \ge (1-\varepsilon)n$ and a shift K_u such that

$$\frac{1}{\sqrt{n}}\ell(K_u\cap E) \prec n^{1/3}\log^6 d_K \prec n^{1/3}.$$

Let $F = E^{\perp}$. Then $\dim F = C\log n$ and by Lemma 4 there exists a shift K_v such that

$$\ell(K_v\cap F) \le (C\log n)^2\cdot\ell(K-K) \le (C\log n)^2\cdot C\sqrt{n}\cdot\log d_K \prec \sqrt{n}.$$

Put $w = (u+v)/2$. Then

$$\ell(K_w) \leq \ell(K_w \cap E) + \ell(K_w \cap F) \leq 2\ell(K_u \cap E) + 2\ell(K_v \cap F) \prec n^{5/6} + n^{1/2} \prec n^{5/6}.$$

Since

$$\ell(K_w^\circ) = \ell(K^\circ) \leq \sqrt{n},$$

this completes the proof of Theorem 2. □

The proof of Theorem 4 follows closely the standard iteration argument [P, Chapter 8], so we shall only sketch it.

Proof of Theorem 4 (Sketch). There exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that the body $TK + x$ satisfies the estimates of Theorem 1. Obviously, it is enough to prove Theorem 4 for TK instead of K, so we shall assume that $T = id$.

Let Q_x be the orthogonal projection with Ker $Q_x = \mathrm{span}(x)$. Notice that it is enough to construct a projection of a section through 0 of the body $\bar{K} = Q_x K$. For this body we have

$$M^*(\bar{K}) \leq 2M^*(K) \leq 2$$

and for any $\varepsilon > 0$ there exists a linear subspace E of dimension at least $(1-\varepsilon)n$ such that

$$M(\bar{K} \cap E) \prec \frac{1}{\varepsilon} \log d_{\bar{K}}.$$

It follows from [P, Theorem 5.8] that there exists a linear subspace $V \subset E$ of dimension bigger than $(1-2\varepsilon)n$ such that

$$\bar{K} \cap V \subset (\bar{K} - \bar{K}) \cap V \subset \lambda \cdot (B_2^n \cap V), \tag{9}$$

where

$$\lambda = \frac{C}{\varepsilon^{1/2}} M^*(\bar{K} - \bar{K}) \leq \frac{C'}{\varepsilon^{1/2}} M^*(\bar{K}) \leq \frac{C''}{\varepsilon^{1/2}}.$$

Applying [P, Theorem 5.8] again, we show that there exists a further subspace $W \subset V$ such that dim $W > (1-3\varepsilon)n$ and

$$P_W(\bar{K} \cap V) \supset P_W((\bar{K} \cap (-\bar{K})) \cap V) \supset \mu^{-1} \cdot (B_2^n \cap W), \tag{10}$$

where

$$\mu = \frac{C}{\varepsilon^{1/2}} M(\bar{K} \cap (-\bar{K})) \leq \frac{C}{\varepsilon^{1/2}} M(\bar{K}) \prec \frac{1}{\varepsilon^{3/2}} \log d_{\bar{K}}.$$

Finally, it follows from (9) and (10) that

$$d_{P_W}(\bar{K} \cap V) \prec \frac{1}{\varepsilon^2} \log d_{\bar{K}}.$$

Theorem 4 follows now from Milman's iteration argument similar to [P, Lemmas 8.5 and 8.6]. □

5. Distances between convex bodies.

We shall use the following Theorem due to Benyaminy and Gordon [B-G].

Theorem 6. *Let K and D be n-dimensional convex bodies. then*

$$d(K,D) \le \frac{C}{n} \cdot \Big(\|id : B_2^n \to K^\circ\| \cdot \ell(D) + \|id : B_2^n \to D\| \cdot \ell(K^\circ) \Big) \times \Big(\|id : B_2^n \to D^\circ\| \cdot \ell(K) + \|id : B_2^n \to K\| \cdot \ell(D^\circ) \Big).$$

Notice that although Theorem 6 was proved under the assumption that the bodies K and D are symmetric, the same proof works for general convex bodies.

By a standard contraction argument

$$\|id : B_2^n \to K\| \le \ell(K),$$

so by Theorem 2 we have

$$d(K,D) \le \frac{C}{n} \cdot \ell(K^\circ)\ell(D) \cdot \ell(D^\circ)\ell(K).$$

However combining this approach with a result of Banasczyk, Litvak, Pajor and Szarek [B-L-P-S, Proposition 3.1 and Remark 3.2] (see also [B]), we obtain a better estimate. More precisely, we need the following

Theorem 7. *Let K be an n-dimensional convex body and let B_2^n be the ellipsoid of maximal volume inscribed in K. Then*

$$\ell(K^\circ) \le Cn\sqrt{\log n}.$$

Proof of Theorem 5. Let $W = \max \Big(M(K)M^*(K), M(D)M^*(D) \Big)$. It is enough to prove that

$$d(K,D) \le Cn \cdot \sqrt{\log n} \cdot W.$$

Assume that the body K is embedded into $\mathbb{R}^n$ so that

$$\ell(K) \le \sqrt{n} \quad \text{end} \quad \ell(K^\circ) \le \sqrt{n} \cdot W.$$

Let S be a linear operator which maps the ellipsoid of maximal volume inscribed in K onto B_2^n. Put

$$T = id + \frac{W}{\sqrt{n \log n}} S.$$

Then

$$\|id : B_2^n \to TK\| \le \left\| id : B_2^n \to \frac{W}{\sqrt{n\log n}} SK \right\| \le \frac{\sqrt{n \log n}}{W}. \tag{11}$$

Also we have

$$\ell(TK) \le \ell(K) \le \sqrt{n} \tag{12}$$

and by Theorem 7

$$\begin{aligned} \ell((TK)^\circ) &\le \ell(K^\circ) + \frac{W}{\sqrt{n \log n}} \cdot \ell((SK)^\circ) \\ &\le \sqrt{n} \cdot W + \frac{W}{\sqrt{n \log n}} \cdot Cn\sqrt{\log n} \le C\sqrt{n} \cdot W. \end{aligned} \tag{13}$$

By the contraction argument

$$\|id : B_2^n \to (TK)^\circ\| \le \ell((TK)^\circ) \le C\sqrt{n} \cdot W. \tag{14}$$

Similarly, there exists an embedding U of D into $\mathbb{R}^n$ such that

$$\|id : B_2^n \to (UD)^\circ\| \le \frac{\sqrt{n \log n}}{W}, \qquad \ell((UD)^\circ) \le \sqrt{n}$$

and

$$\|id : B_2^n \to UD\| \le \ell(UD) \le C\sqrt{n} \cdot W.$$

Substituting these estimates and (11) – (14) into Theorem 6 we obtain the statement of Theorem 5. □

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